APPROXIMATE CALCULATION OF OPTIMAL CONTROL PROBLEMS

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An averaging method for solving a broad class of optimal control problems for systems reducible to the standard form of systems with a rapidly rotating phase [1] is proposed. The variant of the averaging method used here was proposed by N. N. Bogoliubov and D. N. Zubarev as an extension of the Krylov-Bogoliubov method. A general approximate calculation scheme is described and several variational problems are solved as examples. The proposed approach turns out to be an effective tool for the approximate synthesis of optimal systems.

1. Preliminary note. Let a controlled process be described by a system of n differential equations

$$dx/dt = f(x, u) \tag{1.1}$$

Here t is the time, $x = (x_1, \ldots, x_n)$ is an n-dimensional vector of the phase coordinates, u is the control (a scalar function), and $f = (f_1, \ldots, f_n)$ is a given n-dimensional vector function.

The instant of termination T of the process is fixed. Without limiting generality we can assume that the functional to be minimized is $x_n(T)$. The system is subject to the initial conditions

$$x(t_0) = x_0 \tag{1.2}$$

Here and below the zero subscript denotes the initial value of the function. The conditions at the end of motion are defined by k relations

$$\varphi(x(T)) = 0 \tag{1.3}$$

Here $\varphi = (\varphi_1, \ldots, \varphi_k)$ is a given k-dimensional vector function $(1 \leqslant k < n)$.

The problem of determining the optimal control can be formulated as follows: we are to find the control $u(t)$ and the corresponding optimal trajectory $x(t)$ which satisfy Eq. (1.1), conditions (1.2), (1.3), and the restrictions imposed on the control $u(t) \in V$ for $t_0 \leqslant t \leqslant T$, and then minimize the functional $x_n(T)$. Here V is a closed set.

Let us apply the maximum principle [2] to this problem. We introduce the n-dimensional vector of associated variables (moments) $p = (p_1, \ldots, p_n)$ and write the Hamiltonian H, system (1.1), and the equations for the associated variables as

$$H(x, p, u) = (p, f) \tag{1.4}$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \tag{1.5}$$

According to the maximum principle the control u_* can be determined from the maximum condition for the function H with respect to u, i. e.

$$H(x(t), p(t), u_*) = \max H(x(t), p(t), u), \qquad u \in V \tag{1.6}$$

where

$$p_n(T) < 0 \tag{1.7}$$

Conditions (1.2), (1.3) together with the $n - k$ transversality conditions constitute the complete set of boundary conditions for system (1.5).

We assume that the optimal solution $u_*(x, p)$ is such that the corresponding solution of the system (1.5) "pierces" the surface of discontinuities of the function $u_*(x, p)$.

Let us denote the value of the Hamiltonian for the control chosen from the maximum condition $U(x, p) = H(x, p, u_*)$. This enables us to express system (1.5) in canonical form, using U as our generating function,

$$\frac{dx}{dt} = \frac{\partial U}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial U}{\partial x} \tag{1.8}$$

In fact, assuming that the function H is continuously differentiable with respect to u and that the function u_* has piecewise-continuous partial derivatives with respect to all its arguments, we can write out the following detailed expressions for the right sides of Eqs. (1.8);

$$\frac{\partial U}{\partial p} = \frac{\partial H}{\partial p} + \frac{\partial H}{\partial u}\frac{\partial u}{\partial p}, \qquad \frac{\partial U}{\partial x} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} \tag{1.9}$$

Here we assume that $u = u_*$ was set into the expression for the function H once the derivatives with respect to x, p, u had been computed. In those cases where the maximum of H with respect to u is attained inside the set of permissible controls V we have $\partial H / \partial u = 0$; if the maximum is attained on the boundary of V, then $\partial u_* / \partial x = \partial u_* / \partial p = 0$, since the boundary of the set V does not depend on x, p. This implies that along the trajectories associated with the control determined from (1.7) we have the equation

$$\frac{\partial H}{\partial p} = \frac{\partial U}{\partial p}, \qquad \frac{\partial H}{\partial x} = \frac{\partial U}{\partial x}$$

so that we can use system (1.8) instead of (1.5). Further on this fact will enable us to simplify the approximate calculations considerably.

2. Asymptotic integration. Let us assume that system (1.1) has been reduced to the standard form of systems with rapidly rotating phases [1],

$$dx / dt = \varepsilon f(x, y, u), \qquad dy / dt = \omega + \varepsilon F(x, y, u) \tag{2.1}$$

Here ε is a small parameter ($\varepsilon \ll 1$), $y = (y_1, \ldots, y_s)$ is the s-dimensional phase vector; $\omega = (\omega_1, \ldots, \omega_s)$ is a set of s frequencies which we assume to be constant; $F = (F_1, \ldots, F_s)$ is a given s-dimensional vector function. The meaning of the functions x, u, f, φ remains unchanged, but f, φ now depend on x, y, u.

The n-dimensional vector x in system (2.1) varies slowly, since its derivative is proportional to the small parameter ε; the vector y varies rapidly, since all of the frequencies $\omega_i \sim 1$.

Let us write out the initial and boundary conditions for (2.1),

$$x(t_0) = x_0, \; y(t_0) = y_0, \qquad \varphi(x, y, T) = 0 \tag{2.2}$$

Let us associate the vectors x, y with the momentum vectors p, λ. The Hamiltonian is of the form

$$H(x, y, p, \lambda, u) = (f, p) + (\lambda, \omega + \varepsilon F) \tag{2.3}$$

Our problem consists in determining the optimal control $u(t)$ and the corresponding trajectory $x(t)$, $y(t)$ which minimize the functional $x_n(T)$ and satisfy Eqs. (2.1), conditions (2.2), and the restrictions imposed on the control.

Having determined the control u_* from the maximum condition for the function H, we construct the function

$$U(x, y, p, \lambda) = H(x, y, p, \lambda, u_*) \tag{2.4}$$

System (2.1) and the adjoint system can be written as

$$\frac{dx}{dt} = \frac{\partial U}{\partial p}, \quad \frac{dy}{dt} = \frac{\partial U}{\partial \lambda}, \quad \frac{dp}{dt} = -\frac{\partial U}{\partial x}, \quad \frac{d\lambda}{dt} = -\frac{\partial U}{\partial y} \tag{2.5}$$

The function U is constant along solutions (2.5) by virtue of the autonomy of system (2.1).

The resulting system (2.5) once again has the standard form of systems with rapidly rotating phases; it is characterized by slow variation of x, p, λ and rapid variation of the vector y.

We can solve the boundary value problem for system (2.5) approximately with the aid of the averaging method. First let us consider the nonresonance case where the frequencies ω_i are noncommensurate. The first-approximation system for (2.5) can be obtained by independent averaging of the right sides of the system over the components of the vector y. From the form of system (2.5) we infer that we can begin by averaging the function U,

$$U_* = \lim_T \frac{1}{T_1 \ldots T_s} \int_0^{T_1} \cdots \int_0^{T_s} U dy_1 dy_2 \ldots dy_s \tag{2.6}$$

$$T_1 \to \infty, \ldots, T_s \to \infty$$

Then, substituting U_* into (2.5), we obtain

$$\frac{dx}{dt} = \frac{\partial U_*}{\partial p}, \quad \frac{dy}{dt} = \frac{\partial U_*}{\partial \lambda}, \quad \frac{dp}{dt} = -\frac{\partial U_*}{\partial x}, \quad \frac{d\lambda}{dt} = -\frac{\partial U_*}{\partial y} \tag{2.7}$$

Boundary conditions (2.2) remain unchanged. If the function U is periodic with the periods $T_1, \ldots, T_s$ in the components $y_1, \ldots, y_s$ of the phase vector, then formula assumes the simpler form

$$U_* = \frac{1}{T_1 \cdots T_s} \int_0^{T_1} \cdots \int_0^{T_s} U\, dy_1 \ldots dy_s \tag{2.8}$$

In computing integrals (2.6), (2.8) we assume that the slowly varying parameters x, p, λ remain constant.

If all the conditions of applicability of asymptotic integration theorems [1, 3] are fulfilled for system (2.5), then the solution of the boundary value problem for system (2.7) approximates the exact solution of the boundary value problem for system (2.5) to within $\sim\varepsilon$ in the interval $T \sim \varepsilon^{-1}$. More precisely, the solution $x^{(1)}(t)$ obtained from (2.5) and the solution $x^{(2)}(t)$ obtained from (2.7), for example, satisfy the relation

$$\max | x^{(1)}(t) - x^{(2)}(t) | = O(\varepsilon) \qquad (0 \leqslant t \leqslant T)$$

Thus, having solved (2.7), we obtain the optimal trajectory and momenta in the first approximation; substituting the solutions into the function u_*, we obtain the control. Along the trajectory associated with this control, the value of the functional differs from the exact value by $\sim \varepsilon$.

Averaged system (2.7) is much simpler than initial system (2.5), since its right side does not contain the phase vector y. In some cases it is possible to integrate system (2.7) completely, to express the control as a function of the phase coordinates x, y, and thus to solve the synthesis problem in the first approximation. In more complicated problems it is necessary to solve (2.7) numerically, but here too it is easier to integrate (2.7) than it is to integrate (2.5) because of the larger integration interval possible in the case of the former.

System (2.7) has the first integrals

$$U_* = \text{const}, \qquad \lambda = \lambda_0, \qquad y = y_0 + \int_0^t \frac{\partial U_*}{\partial \lambda} dt \tag{2.9}$$

Let us write out the formulas for calculating the first approximation in the simplest case where $n = s = 1$. From the maximum condition

$$H = \varepsilon p f + \lambda (\omega + \varepsilon F) \tag{2.10}$$

we obtain u^* and then

$$U_* = pf_* + \lambda(\omega + \varepsilon F_*)$$

$$f_* = \lim_{T\to\infty} \frac{1}{T}\int_0^T f\,dt, \qquad F_* = \lim_{T\to\infty} \frac{1}{T}\int_0^T F\,dt$$

From system (2.7) and first integrals (2.9) we obtain

$$\frac{dx}{dt} = \varepsilon f_*, \quad \lambda = \lambda_0, \quad y = y_0 + \omega t + \varepsilon\int_0^t F_* dt, \quad U_* = \mathrm{const} \tag{2.11}$$

Having determined the function p from the first integral U_*, we reduce solution of the initial problem to the quadrature

$$t = \frac{1}{\varepsilon}\int_{x_0}^{x} \frac{1}{f_*}\,dx \tag{2.12}$$

If we are solving a problem with a free right end (i.e. if conditions (1.3) do not apply) then

$$\lambda = 0, \qquad U_* = \varepsilon\, p\,(t) f_*\,(t) = \varepsilon p\,(T) f_*\,(T)$$

Let us suppose that the function $f_*\,(t)$ does not change sign for $t_0 \leqslant t \leqslant T$; then, by (1.7), $p\,(t) < 0$. The control obtained by means of the maximum principle then coincides with the control obtained from the minimum condition for the derivative dx/dt. The resulting control u_* can then be found immediately in the form of functions of the phase coordinates x, y, i.e. in the form of a synthesis. This procedure for finding the control is used extensively in celestial mechanics (see the survey and bibliography in [4]) and is called "local optimality". The above results show that locally optimal trajectories are close to the optimal ones in the simplest cases such as those considered in Sects. 3 and 4. Generally the control is more complicated even in the first approximation. The control in the problem of Sect. 5, for example, cannot be found from the local optimality condition.

We say that a system is subject to resonance effects if at least two of the frequencies ω_i, ω_j are commensurate, i.e. if there exist relatively prime integers m, l such that

$$\omega_i - \frac{m}{l}\,\omega_j \sim O(\varepsilon) \tag{2.13}$$

The computation scheme is somewhat different in this case. For simplicity we assume that only the two frequencies of (2.13) are commensurate. In place of y_i we introduce the new variable x_{n+1} (the phase shift) by way of the relation

$$x_{n+1} = y_i - \frac{m}{l}\,y_j \tag{2.14}$$

Next, we reduce system (2.1) to the standard form

$$\begin{aligned}
&\frac{dx}{dt} = \varepsilon f(x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_s, x_{n+1}, u)\\
&\frac{dx_{n+1}}{dt} = \omega_i - \frac{m}{l}\,\omega_j + \varepsilon\left(F_i - \frac{m}{l}\,F_j\right)\\
&\frac{dy_1}{dt} = \omega_1 + \varepsilon F_1(x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_s, x_{n+1}, u)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{dy_{i-1}}{dt} = \omega_{i-1} + \varepsilon F_{i-1}
\end{aligned} \tag{2.15}$$

$$\frac{dy_{i+1}}{dt} = \omega_{i+1} + \varepsilon F_{i+1}$$

$$\dots\dots\dots\dots$$

$$\frac{dy_s}{dt} = \omega_s + \varepsilon F_s$$

By condition (2.13), the quantities x, x_{n+1} in these expressions vary slowly, while the $s - 1$ phases $y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_s$ vary rapidly. Next, we apply the maximum principle to (2.15) and average independently over the $s - 1$ components of the phase vector, assuming that the slowly varying parameters are constant.

For some m and l it may turn out that solution (2.15) does not differ from the solution of system (2.7) in a nonresonance case. We then say that there are no resonance effects for these m and l (even though resonance conditions (2.13) are fulfilled).

The problem is more complicated if the frequencies depend on the slowly varying parameters, i.e. if $\omega = \omega(x)$. The derivatives of some of the momenta in this case are approximately equal to unity. This gives rise to additional resonance states; moreover, resonance conditions (2.13) can vary in the course of motion. Averaging entails the same difficulties as does passage through resonance zones in problems on nonlinear oscillations [5, 6].

The above formulas enable us to solve the problem in the first approximation. Higher-order approximations can be obtained by the standard asymptotic integration procedure [1].

Let us consider three very simple problems to illustrate our method.

3. The problem of optimal parametric excitation. Let some oscillatory system be described by the equation

$$d^2z / d\tau^2 + (1 - \varepsilon u)z = 0 \tag{3.1}$$

Here u is a control subject to the condition $0 \leqslant u \leqslant 1$. Of all the possible control laws $u(\tau)$ in the interval $\tau_0 \leqslant \tau \leqslant T$ we must find one for which the energy integral

$$h = {}^1/_2\,(z^2 + z^{\cdot 2}) \tag{3.2}$$

assumes its smallest (largest) value at the end of the process.

By the standard substitution of variables

$$z = x \cos y, \qquad dz/d\tau = - x \sin y \tag{3.3}$$

we reduce (3.1) and (3.2) to the standard form

$$x^{\cdot} = - \varepsilon ux \cos y \sin y, \qquad y^{\cdot} = 1 - \varepsilon u \cos^2 y \tag{3.4}$$

$$h = 0.5\, x^2 \tag{3.5}$$

We infer from (3.5) that the initial problem is equivalent to the minimization (maximization) of the amplitude x. From the maximum condition

$$H = \lambda - \varepsilon\, u \cos^2 y\, [\lambda + xp \operatorname{tg} y] \tag{3.6}$$

we obtain

$$u = \theta\,(- \lambda - xp \operatorname{tg} y) \tag{3.7}$$

Here we have introduced the Heaviside function: $\theta(z) = 0$ for $z < 0$ and $\theta(z) = 1$ for $z > 0$.

The boundary conditions are of the form

$$x(\tau_0) = x_0, \qquad y(\tau_0) = y_0, \qquad p(T) = -1, \;\; \lambda(T) = 0 \tag{3.8}$$

Substituting (3.7) into (3.6), we obtain

$$U = \lambda - \varepsilon\,[\lambda\cos^2 y + xp\cos y\sin y]\,\theta\,[-\lambda - xp\,\mathrm{tg}\,y]$$

After averaging we have

$$U_* = \lambda - \frac{\varepsilon}{2\pi}\left[xp + \frac{\lambda\pi}{2} + \lambda\,\mathrm{arc\,tg}\,\frac{\lambda}{xp}\right]$$

Making use of formulas (2.11) with allowance for (3.8), we obtain

$$\lambda = 0,\ p = -\frac{x(T)}{x(\tau)},\ x = x_0\exp\frac{-\varepsilon(\tau-\tau_0)}{2\pi},\ y = y_0 + \left(1-\frac{\varepsilon}{4}\right)(\tau-\tau_0) \quad (3.9)$$

Substituting (3.9) into (3.7), we obtain the optimal (first-approximation) control law

$$u = \theta\,(\pm\,\mathrm{tg}\,y) = \theta\,(\mp\, y\dot{y})$$

Here the upper signs correspond to the optimal decrease in the oscillation amplitude and the lower signs to the optimal increase.

4. Rotational motion. The proposed scheme can be used to compute the optimal controls for essentially nonlinear systems. For example, let us consider the problem of the optimal decrease of energy of a rotating pendulum. The equation of motion can be written as

$$d^2y/dt^2 + w\sin y = 0 \quad (4.1)$$

We must choose the control $w(t)$ from the interval $w_2 \leqslant w \leqslant w_1$ in such a way that the angular velocity assumes its smallest value at the end of motion. We assume that the pendulum rotates rapidly throughout the process. Following [7], we introduce the new variables x, the time τ, and the initial angular velocity Ω by means of the relations

$$dy/dt = \Omega + x, \qquad \tau = \varepsilon^{-1}t, \qquad \Omega = \varepsilon^{-1}$$

where $\varepsilon \ll 1$ is a small parameter. We obtain a system equivalent to Eq. (4.1),

$$dx/d\tau = -\varepsilon w\sin y, \qquad dy/d\tau = 1 + \varepsilon x$$

The boundary conditions for the optimal problem are as follows:

$$x(\tau_0) = 0, \quad y(\tau_0) = \Omega, \quad p(T) = -1, \quad \lambda(T) = 0$$

Let us write out the function H, the optimal law of variation of w, and the averaged function U_*,

$$H = \lambda + \varepsilon x\lambda - \varepsilon pw\sin y$$

$$w = w_1\,\theta\,(-p\sin y) + w_2\,\theta\,(p\sin y)$$

$$U_* = \lambda(1+\varepsilon x) + \frac{\varepsilon p}{\pi}(w_1 - w_2)$$

The final solution of the problem is of the form

$$x = \frac{(w_2 - w_1)}{\pi}(t-t_0), \quad y = \Omega + \Omega(t-t_0) + \frac{(t-t_0)^2}{2\pi}(w_2 - w_1) + O(\Omega^{-1})$$

5. Stabilization of motion. The problem of choosing the optimal control law for the stabilization of an oscillatory system acted on by perturbing forces arises in various problems of the theory of nonlinear oscillations. The dynamics of such processes for the simplest stabilization laws has been investigated by several authors. The averaging method appears to be an exceptionally effective means of solving problems of synthesis of stabilizing systems. For example, let us solve the problem of stabilization of the relative motion of a satellite in a near-circular orbit.

We assume that the satellite is equipped with two low-thrust engines which generate a certain controlling moment D. Let the principal central axis of inertia of the satellite be perpendicular to the orbital plane at all times; the moment of inertia about this axis is equal to B. We denote the moment of inertia about the two other axes by A and C $(A > C)$.

We describe the relative motion by means of the angle φ – the angle between the direction of the orbital perigee and the principal central axis of inertia of the satellite associated with the moment of inertia C. To within quantities on the order of the ratio of the satellite dimensions to the dimensions of the orbit the equation of relative motion is of the form [8]

$$\frac{d^2\varphi}{d\tau^2} = \frac{E^2}{2}(1 + e\cos\vartheta)^3 \sin 2(\alpha + \gamma) - \varkappa u \tag{5.1}$$

$$3 \geqslant E^2 = \frac{3(C-A)}{B}, \quad \varkappa = \frac{DP^3}{B\mu}, \quad \tau = t\sqrt{\mu}\, P^{-3/2}, \quad u = \frac{D}{D_1}$$

Here e is the eccentricity of the orbit, ϑ is the true anomaly, P is the focal parameter of the orbit, D_1 is a certain characteristic parameter of the controlling moment, $^1/_2\,\pi - \alpha - \gamma$ is the angle between the radius vector of the center of mass and the axis of inertia of the satellite associated with the moment of inertia C, and α is the angle between the velocity vector and the direction normal to the radius vector of the center of mass. We have the formula

$$\sin\alpha = \frac{e\sin\vartheta}{\sqrt{1 + 2e\cos\vartheta + e^2}} \tag{5.2}$$

The angles introduced above are related by the expression

$$\gamma = {}^1/_2\pi + \vartheta - \alpha - \varphi \tag{5.3}$$

Let us assume that e and $\varkappa$ are small quantities of the same order of magnitude. Then, neglecting the evolution of the center-of-mass orbit of the satellite, we obtain

$$\alpha = e\sin\vartheta + O(e^2), \qquad \frac{d\vartheta}{d\tau} = 1 + 2e\cos\vartheta + O(e^2) \tag{5.4}$$

Linearizing Eq. (5.1) for small γ with allowance for (5.3), (5.4), we obtain

$$\frac{d^2\gamma}{d\tau^2} + E^2\gamma = \varkappa u - e\,[(1 + E^2)\sin\vartheta + 3\gamma E^2\cos\vartheta] = -\psi \tag{5.5}$$

Following [5], we formulate the problem of optimal stabilization with respect to the motion of the satellite as follows: we are required to synthesize the control $u(\gamma, \gamma^{\cdot}, \tau)$ in such a way that the functional

$$J = \varkappa\int_0^T (u^2 + c^2\gamma^2)\, d\tau \tag{5.6}$$

assumes its minimum value in a fixed time of motion T. Here c is some constant. The first term in the integrand is the integral penalty for the large value of the controlling moment; the second term is the integral penalty for the large deviation of the angle γ.

Let us introduce the new variables x_1, x_2, y by means of the expressions

$$\gamma = x_1\cos y, \quad \frac{d\gamma}{d\tau} = -Ex_1\sin y, \quad \frac{dx_2}{d\tau} = \varkappa(u^2 + c^2\gamma^2) \tag{5.7}$$

The equations of relative motion and of the center-of-mass motion written in the standard form are

$$\frac{dx_1}{d\tau} = \frac{\psi}{E}\sin y, \qquad \frac{dx_2}{d\tau} = \varkappa(u^2 + c^2x_1^2\cos^2 y)$$
$$\frac{dy}{d\tau} = E + \frac{\psi}{Ex_1}\cos y, \qquad \frac{d\vartheta}{d\tau} = 1 + 2e\cos\vartheta \tag{5.8}$$

Here x_1, x_2 vary slowly; the rapidly varying parameters are y, ϑ. We associate the functions x_1, x_2, y, ϑ with the momenta $p_1, p_2, \lambda_1, \lambda_2$.

The boundary conditions for the variational problem are

$$x_1(0) = x_0, \quad y(0) = y_0, \; p_2(T) = -1$$
$$x_2(0) = p_1(T) = \lambda_1(T) = \lambda_2(T) = 0 \tag{5.9}$$

The minimizing functional is $J = x_2(T)$. Let us write out the function H, the optimal control, and the function U for the nonresonance case,

$$H = \frac{\psi p_1}{E}\sin y + \lambda_1\left[E + \frac{\psi\cos y}{Ex_1}\right] + xp_2(u^2 + c^2x_1^2\cos^2 y) + \lambda_2(1 + 2e\cos\vartheta)$$
$$u = \frac{1}{2Ep_2}\left[p_1\sin y + \frac{\lambda_1\cos y}{x_1}\right] \tag{5.10}$$

$$U = -\frac{\varkappa}{4E^2p_2}\left[p_1\sin y + \frac{\lambda_1\cos y}{x_1}\right]^2 + \frac{e}{E}\left(p_1\sin y + \frac{\lambda_1\cos y}{x_1}\right)[(1+E^2)\sin\vartheta +$$
$$+ 3x_1E^2\cos y\cos\vartheta] + \varkappa p_2c^2x_1^2\cos^2 y + \lambda_2(1 + 2e\cos\vartheta) + \lambda_1 E$$

Averaging and making use of (2. 9), (5. 9), we obtain

$$U_* = -\frac{\varkappa}{8E^2p_2}\left[p_1^2 + \frac{\lambda_1^2}{x_1^2}\right] + \frac{\varkappa}{2}p_2c^2x_1^2 + E\lambda_1 + \lambda_2 \tag{5.11}$$
$$\lambda_1 = \lambda_2 = 0, \; p_2 = -1, \quad y = y_0 + E\tau, \quad \vartheta = \vartheta_0 + \tau$$

Integrating the system

$$\frac{dx_1}{d\tau} = \frac{\varkappa p_1}{4E^2}, \quad \frac{dx_2}{d\tau} = \frac{\varkappa}{2}\left[c^2x_1^2 + \frac{p_1^2}{4E^2}\right], \quad \frac{dp_1}{d\tau} = \varkappa x_1 c^2 \tag{5.12}$$

we obtain

$$x = x_{10}[\operatorname{ch}\tau k + \Phi\operatorname{sh}\tau k], \qquad p_1 = p_{10}\left[\operatorname{ch}\tau k + \frac{\operatorname{sh}\tau k}{\Phi}\right]$$
$$J = \frac{\varkappa c^2x_{10}^2}{2k}\operatorname{sh}^2 kT\,[2\Phi + (1+\Phi^2)\operatorname{cth}kT] \tag{5.13}$$
$$k = \frac{\varkappa c}{2E}, \qquad \Phi = \frac{p_{10}}{2Ecx_{10}}$$

Conditions (5. 9) yield

$$p_{10} = -2cx_{10}\operatorname{th}kT \tag{5.14}$$

Substituting (5. 14) into (5. 10), (5. 13), we obtain

$$x = x_{10}\frac{\operatorname{ch}k(T-\tau)}{\operatorname{ch}kT}, \quad J = cx_{10}^2\operatorname{th}kT, \quad p = -2cx\operatorname{th}k(T-\tau)$$

From this solution we conclude that the amplitude of the satellite oscillations about the velocity vector decreases monotonically and that the controlling moment oscillates at the frequency of the relative motion and with a slowly decreasing amplitude equal to zero for $\tau = T$. Expressing u in terms of the phase coordinates, we obtain the solution of the optimal correction synthesis problem,

$$u = -c\frac{d\gamma}{d\tau}\operatorname{th}k(T-\tau) \tag{5.15}$$

System (5. 8) has resonance effects for $E = 1$, $E = 0.5$. These can be investigated by the scheme of Sect. 2.

The controls obtained in the above examples can differ from the exact optimal controls by ~ 1. However, the approximate values of the functionals and phase coordinates approximate the exact ones to within $\sim \varepsilon$, $\varkappa$ in the interval of motion $T \sim \varepsilon^{-1}$, $\varkappa^{-1}$ respectively. In this sense the above controls are optimal in the first approximation.

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STABILITY OF THE PERIODIC SOLUTIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

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Sufficient conditions for the asymptotic stability of quasilinear autonomous systems consisting of second-order equations are derived. The generating systems can have simple and multiple, commensurate and partly noncommensurate, and zero frequencies. The investigation is carried out with the aid of equations in variations for sufficiently small values of the parameter μ.

1. Let us consider the following quasilinear autonomous system with n degrees of freedom:

$$\sum_{k=1}^{n} (a_{ik}x''_k + c_{ik}x_k) = \mu F_i(x_1, \ldots, x_n, x'_1, \ldots, x'_n, \mu)$$

$$a_{ik} = a_{ki}, \qquad c_{ik} = c_{ki} \qquad (i = 1, \ldots, n)$$